

# Long term behaviour of locally interacting birth-and-death processes

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## Abstract

In this paper we study long-term evolution of a finite system of locally interacting birth-and-death processes labelled by vertices of a finite connected graph. A detailed description of the asymptotic behaviour is obtained in the case of both constant vertex degree graphs and star graphs. The model is motivated by modelling interactions between populations and is related to interacting particle systems, Gibbs models with unbounded spins, as well as urn models with interaction.

## 1 The model

Let  $\Lambda$  be a finite connected graph. If two vertices  $x, y \in \Lambda$  are connected by an edge, call them *neighbours* and write  $x \sim y$ . Let  $\mathbb{Z}$  be the set of all integers and  $\mathbb{Z}_+$  be the set of all non-negative integers including zero. Consider a continuous time Markov chain (CTMC)  $\xi(t) = \{\xi_x(t), x \in \Lambda\} \in \mathbb{Z}_+^\Lambda$  with the following transition rates: given  $\xi(t) = \xi \in \mathbb{Z}_+^\Lambda$  a component (a spin)  $\xi_x$  increases by 1 at the rate  $e^{\alpha\xi_x + \beta\phi(x, \xi)}$ , where  $\alpha, \beta \in \mathbb{R}$ ,

$$\phi(x, \xi) = \sum_{y: y \sim x} \xi_y \quad (1)$$

and at the same time each positive component  $\xi_x$  decreases by 1 at constant rate 1.

This birth-and-death dynamics belongs to a class of stochastic dynamics which is used in statistical physics to describe the time evolution of a system of interacting spins. Our particular dynamics is motivated by adsorption-desorption processes, where adsorption rates depend on a local environment and an adsorbed particle can depart at a non-zero rate ([2]). It is closely related to a particle deposition on a discrete substrate and urn models with interaction (e.g., [6], [12], and [13]). Recall also that a birth-and-death process on the non-negative integer half-line is a classic probabilistic model for the population size so that the Markov chain can be used for modelling different types of interaction between populations, where a component  $\xi_x(t)$  can be interpreted as the size of a population which is located at  $x \in \Lambda$  at time  $t$ .

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If we assume that spins are bounded and consider the same birth-and-death dynamics then we will get a finite ergodic Markov chain whose equilibrium distribution is a Gibbs measure (see Remark 1). A particular case of the model with bounded spins, where  $\alpha = \beta$ ,  $\Lambda \subset \mathbb{Z}^d$ , was studied in [14]. For instance, if a spin takes values 0 and 1 only, and, in addition,  $\alpha = \beta > 0$ , then we obtain a finite Markov chain whose equilibrium distribution is a Gibbs measure on  $\{0, 1\}^\Lambda$  which is equivalent to a particular case of the famous Ising model on  $\{-1, 1\}^\Lambda$ . The main goal in [14] was to study the asymptotic behaviour of the stationary distribution as  $\Lambda \uparrow \mathbb{Z}^d$ . In general, the asymptotic behaviour of such equilibrium distributions in thermodynamic limit, i.e. as graph  $\Lambda$  expands, is of main interest in statistical physics.

The aim of this paper, on the other hand, is to describe the asymptotic behaviour of the Markov chain with *unbounded* spins as time tends to infinity while the underlying graph remains fixed. In this case we deal with a countable Markov chain that can be either recurrent (or even ergodic) or non-recurrent (e.g., transient, or even explosive) depending both on the graph  $\Lambda$  and the values of parameters  $\alpha, \beta$ .

It is easy to see that if  $\beta = 0$  then the structure of graph  $\Lambda$  is irrelevant and the components of CTMC  $\xi(t)$  are independent identically distributed birth-and-death processes with values in  $\mathbb{Z}_+$ . The well known results for birth-and-death processes (e.g. see [3] or [11]) yield that if  $\alpha > 0, \beta = 0$ , then each component is an explosive MC. In turn, it implies that CTMC  $\xi(t)$  is explosive. Moreover, independence of spins imply that their times to explosion are also independent and this allows to repeat the well known Rubin's argument (used in [1] in the case of classic Polya urn scheme) in order to obtain that with probability 1 only a single component of  $\xi(t)$  explodes. Notice that this fact can be also inferred from our Theorem 2. A non-zero interaction does not change the explosive behaviour of the Markov chain in the case  $\alpha > 0$  but escape to infinity can happen in various ways which depend on both  $\beta$  and  $\Lambda$ .

If  $\alpha < 0, \beta = 0$ , then CMTC  $\xi(t)$  is formed by a collection of independent ergodic Markov chains. It is quite obvious that if both  $\alpha < 0$  and  $\beta < 0$  then the Markov chain remains to be ergodic. If  $\beta > 0$ , then one could intuitively expect that given  $\alpha < 0$  there exists some critical value  $\beta_{cr}$  such that if  $\beta < \beta_{cr}$ , then the stable ergodic evolution of the system is still observed, and, in contrast, if  $\beta > \beta_{cr}$ , then the system becomes unstable, i.e. transient or even explosive. We compute this critical value explicitly in some cases. It turns out that  $\beta_{cr} = -\alpha c(\Lambda)$ , where  $c(\Lambda) = \nu^{-1}$  in the case of a graph  $\Lambda$  with the constant vertex degree  $\nu$  and  $c(\Lambda) = n^{-\frac{1}{2}}$  in the case of a star graph  $\Lambda$  with  $n + 1$  vertices.

The Markov chain under consideration is reversible, therefore the computation of its invariant measure is straightforward. Stationary probability distributions arising in positive recurrent cases are Gibbs measures with unbounded positive spins on a finite graph with empty boundary conditions. Consequently the model in positive recurrent cases is closely related to Gibbs random fields with unbounded spins on graphs (see [5], [7], and references therein).

We give a detailed description of how the Markov chain escapes to infinity in all the transient cases that we consider. We show that due to a rapid increase of birth rates in explosive cases, there are no death events in the system after some finite random moment of time, and the dynamics of the Markov chain is that of a pure birth process, obtained by setting the death rates to zero.

We will start with results that are valid in the case of an arbitrary finite connected graph  $\Lambda$ ; they are presented in Theorems 1, 2 and 3. We also study two special cases in more detail, namely constant vertex degree graphs and star graphs. The results for these two cases are found in Theorems 4, 5 and 6. Graphs with the constant vertex degree and star graphs are particular examples of spatially homogeneous graphs and of spatially inhomogeneous graphs, respectively. Despite the obvious difference in the structure of these graphs the long term behaviour of the corresponding Markov chains is similar to each other. The main features of the model dynamics are illustrated in Section 3 by a model with graph  $\Lambda$  formed by just two neighbouring vertices. Proofs are given in Section 4.

Finally, we denote by  $C_i$ ,  $i = 1, 2, \dots$ , or just  $C$  various constants whose exact values are immaterial and can change from line to line.

## 2 Results

Let  $\Lambda$  be an arbitrary graph. Given  $\xi \in \mathbb{Z}_+^\Lambda$  define *potential*  $U(x, \xi)$  of a vertex  $x \in \Lambda$  as the following quantity

$$U(x, \xi) = \alpha \xi_x + \beta \phi(x, \xi). \quad (2)$$

Notice the following identity

$$\sum_{x \in \Lambda} U(x, \xi) = \sum_{x \in \Lambda} (\alpha + \beta \nu(x)) \xi_x, \quad (3)$$

where  $\nu(x)$  is the degree of vertex  $x \in \Lambda$ , i.e. the number of edges incident to the vertex. Throughout the paper we will also denote by  $1_A$  the indicator of a set (or event)  $A$ . In these notations, given  $\xi(t) = \xi \in \mathbb{Z}_+^\Lambda$  a component  $\xi_x$  jumps up by 1 with intensity  $e^{U(x, \xi)}$  and the generator of the Markov chain is therefore

$$\mathsf{L}f(\xi) = \sum_{x \in \Lambda} (f(\xi + \mathbf{e}^{(x)}) - f(\xi)) e^{U(x, \xi)} + (f(\xi - \mathbf{e}^{(x)}) - f(\xi)) 1_{\{\xi_x > 0\}}, \quad (4)$$

where  $\mathbf{e}^{(x)}$  is a configuration such that  $\mathbf{e}_x^{(x)} = 1$  and  $\mathbf{e}_y^{(x)} = 0$  for all  $y \neq x$  (addition of configurations is understood component-wise).

Recall that the embedded Markov chain, corresponding to a continuous time Markov chain (CTMC), is a discrete time Markov chain (DTMC) with the same state space, and that makes the same jumps as the continuous time Markov chain with probabilities proportional to the corresponding jump rates. Let  $\zeta(t)$  be the DTMC corresponding to CTMC  $\xi(t)$ . The states of the embedded Markov chain will be denoted by  $\zeta$  and we will use the same symbol  $t = 0, 1, 2, \dots$ , to denote the discrete time.

It is easy to see that if  $\beta = 0$  then the components of CTMC  $\xi(t)$  are independent identically distributed birth-and-death processes in  $\mathbb{Z}_+$ . It is easy to see that CTMC  $\xi(t)$  is ergodic, if  $\alpha < 0$ , and is explosive if  $\alpha > 0$  respectively. Also, if both  $\alpha = 0$  and  $\beta = 0$  then CTMC  $\xi(t)$  is formed by a collection of independent simple symmetric random walks in  $\mathbb{Z}_+$  reflected at the origin. This CTMC is null recurrent if  $|\Lambda| = 1, 2$ , and is transient if  $|\Lambda| \geq 3$ . We exclude these trivial cases in what follows.

Let us define the following function

$$W(\xi) = \exp \left( \alpha \sum_x \xi_x (\xi_x - 1)/2 + \beta \sum_{x \sim y} \xi_x \xi_y \right), \quad \xi \in \mathbb{Z}_+^\Lambda. \quad (5)$$

It is easy to see that

$$e^{U(x, \xi)} e^{W(\xi)} = e^{W(\xi + \mathbf{e}^{(x)})}$$

for all  $x \in \Lambda$  and  $\xi \in \mathbb{Z}_+^\Lambda$ . This equation is a detailed balance condition which implies that the Markov chain is time-reversible with invariant measure  $e^{W(\xi)}$ ,  $\xi \in \mathbb{Z}_+^\Lambda$ . According to e.g. Theorem 1.2.4 in [4], an irreducible countable Markov chain is positive recurrent (i.e. ergodic) if and only if there exists a stationary probability distribution, and if the latter exists then the distribution of the Markov chain converges to it as time goes to infinity. Therefore, if

$$Z_{\alpha, \beta, \Lambda} = \sum_{\xi \in \mathbb{Z}_+^\Lambda} e^{W(\xi)} < \infty, \quad (6)$$

then CTMC  $\xi(t)$  is ergodic with the stationary probability distribution given by

$$\mu_{\alpha, \beta, \Lambda}(\xi) = \frac{e^{W(\xi)}}{Z_{\alpha, \beta, \Lambda}}, \quad \xi \in \mathbb{Z}_+^\Lambda. \quad (7)$$

**Remark 1** If a component of the Markov chain takes values in  $\{0, 1, \dots, N\}$ , where  $N \geq 1$ , then the invariant probability distribution of the Markov chain is defined similar to measure (7). Namely, it is a probability measure on  $\{0, 1, \dots, N\}^\Lambda$  that is equal, up to a normalizing constant, to function  $e^{W(\xi)}$ , where, in turn, function  $W$  is defined, as before, by (5).

Denote

$$Q(\xi) = -\alpha \sum_x \xi_x^2 - 2\beta \sum_{x \sim y} \xi_x \xi_y, \quad (8)$$

$$S(\xi) = \sum_x \xi_x. \quad (9)$$

Then we can rewrite function (5) as

$$W(\xi) = -\frac{1}{2}(Q(\xi) + \alpha S(\xi)). \quad (10)$$

Recall that  $\nu(x)$  denotes the degree of vertex  $x \in \Lambda$  and notice the following useful representations of the quadratic part of  $W$

$$\begin{aligned} Q(\xi) &= \sum_x (-\alpha - \beta \nu(x)) \xi_x^2 + \beta \sum_{x \sim y} (\xi_x - \xi_y)^2 \\ &= \sum_{x \in \Lambda} (-\alpha \xi_x^2 - \beta \xi_x \phi(x, \xi)) = - \sum_{x \in \Lambda} \xi_x U(x, \xi). \end{aligned} \quad (11)$$

We are ready now to formulate the findings of our paper. We start with the results that are valid for all finite connected graphs.

**Theorem 1** *Let  $\Lambda$  be a finite connected graph.*

- 1) *If  $\alpha < 0$  and  $\alpha + \beta \max_{x \in \Lambda} \nu(x) \leq 0$  then CTMC  $\xi(t)$  is not explosive. Moreover, if  $\alpha < 0$  and  $\alpha + \beta \max_{x \in \Lambda} \nu(x) < 0$  then CTMC  $\xi(t)$  is ergodic.*
- 2) *If  $\alpha \geq 0$  then CTMC  $\xi(t)$  is not ergodic.*

The transient behaviour of the Markov chain in Part 2) of Theorem 1 can be described more precisely under certain additional assumptions. In order to do so, define the following event related to DTMC  $\zeta(t)$ :

$$B = \{ \exists \tau \in \mathbb{Z}_+ \text{ and a vertex } x \in \Lambda \text{ such that } \zeta_y(\tau + s + 1) = \zeta_y(\tau + s) + 1_{\{y=x\}}, \forall s \geq 1 \}, \quad (12)$$

in other words, the process grows only at point  $x$  after time  $\tau$ .

**Theorem 2** *Let  $\Lambda$  be a finite graph. If  $\alpha > \max\{0, \beta\}$  then with probability 1 event  $B$  defined by (12) occurs, and a single component of CTMC  $\xi(t)$  explodes.*

**Remark 2** Notice that we do not assume connectedness of the underlying graph in Theorem 2.

Furthermore, given  $x_1, x_2 \in \Lambda$  define the following event

$$B_{x_1, x_2} = \{ \exists s \in \mathbb{Z}_+ : \zeta_y(t) = \zeta_y(s) \text{ for all } y \notin \{x_1, x_2\} \text{ and all } t \geq s; \lim_{t \rightarrow \infty} \frac{\zeta_{x_1}(t)}{t} = \lim_{t \rightarrow \infty} \frac{\zeta_{x_2}(t)}{t} = \frac{1}{2} \}. \quad (13)$$

**Theorem 3** *Let  $\Lambda$  be a finite connected graph without triangles, i.e. such that there are no three distinct vertices  $x, y, z \in \Lambda$  such that  $x \sim y$ ,  $y \sim z$  and  $z \sim x$ . If  $0 < \alpha < \beta$  then with probability 1 there are two adjacent vertices  $x_1$  and  $x_2$  such that the event (13) occurs. This implies that with probability 1 only a pair of adjacent components of the CTMC explodes.*

**Theorem 4** *Let  $\Lambda$  be a graph with the constant vertex degree  $\nu(x) \equiv \nu$ .*

- 1) *CTMC  $\xi(t)$  is ergodic if and only if  $\alpha < 0$  and  $\alpha + \beta\nu < 0$ .*
- 2) *If  $\alpha < 0$  and  $\alpha + \beta\nu = 0$  then CTMC  $\xi(t)$  is transient.*
- 3) *If  $\alpha < 0$  and  $\alpha + \beta\nu > 0$  then CTMC  $\xi(t)$  is explosive.*
- 4) *If  $\alpha > 0$  then CTMC  $\xi(t)$  is explosive. Moreover,*
  - i) *if  $\beta < \alpha$  then with probability 1 the event (12) occurs and a single component of CTMC  $\xi(t)$  explodes;*
  - ii) *if  $\alpha < \beta$  and the graph  $\Lambda$  is without triangles (as explained in Theorem 3) then with probability 1 the event  $B_{x_1, x_2}$  occurs for some adjacent vertices  $x_1, x_2 \in \Lambda$ , so that with probability 1 a pair of adjacent components of the CTMC explodes.*

Let us mention two examples of constant vertex degree graphs, both with and without triangles.

- a) *Lattice models with local interaction.* Let  $\mathbb{Z}$  be the set of all integers. Given integers  $L > 0, d \geq 1$ , let  $\Lambda = [-L, -L+1, \dots, 0, \dots, L-1, L]^d \in \mathbb{Z}^d$  be a lattice cube with periodic boundary conditions. Call  $x, y \in \Lambda$  neighbours, if  $|x-y| = 1$ , where  $|\cdot|$  is the Euclidean norm in  $\mathbb{R}^d$ . In this case  $\nu(x) \equiv 2d$  and the graph does not have triangles.
- b) *Mean-field model.* Given  $n \geq 2$  let  $\Lambda$  be a complete graph with  $n$  vertices. By construction,  $\nu(x) \equiv n-1$  in this example and the graph *does have* triangles.

The following statement complements Theorem 4 in the mean field case.

**Theorem 5** *Let  $\Lambda$  be a complete graph with  $n$  vertices labelled by  $1, \dots, n$ , where  $n \geq 1$ . If either  $0 < \alpha < \beta$  or  $\alpha < 0 < \alpha + \beta\nu$  then*

- 1)  $\zeta_k(t)/t \rightarrow 1/n$  for all  $k = 1, \dots, n$  a.s.;
- 2) all components of CTMC  $\xi(t)$  explode simultaneously a.s.;
- 3) a process of differences  $(\zeta_1(t) - \zeta_n(t), \dots, \zeta_{n-1}(t) - \zeta_n(t)) \in \mathbb{Z}^{n-1}$  converges in distribution as  $t \rightarrow \infty$ .

Finally, Theorem 6 below describes the long-term behaviour of the Markov chain in the case of a star graph.

**Theorem 6** *Given  $n \geq 1$  let  $\Lambda$  be a star graph with  $(n+1)$  vertices, i.e. where there is a central vertex  $x$  and its neighbouring vertices  $y_1, \dots, y_n$ , so that  $x$  is the only neighbour for each of  $y_i$ ,  $i = 1, \dots, n$ , and  $x \sim y_i$ ,  $i = 1, \dots, n$ . Then*

- 1) CTMC  $\xi(t)$  is ergodic if and only if  $\alpha < 0$  and  $\alpha + \beta\sqrt{n} < 0$ ;
- 2) if  $\alpha < 0$  and  $\alpha + \beta\sqrt{n} = 0$  then CTMC  $\xi(t)$  is transient;
- 3) if  $\alpha < 0$  and  $\alpha + \beta\sqrt{n} > 0$  then with probability 1

$$\frac{\zeta_x(t)}{t} \rightarrow \frac{n\beta + |\alpha|}{(n+1)\beta + 2|\alpha|}, \quad \frac{\zeta_{y_i}(t)}{t} \rightarrow \frac{\beta + |\alpha|}{(n+1)\beta + 2|\alpha|}, \quad i = 1, 2, \dots, n,$$

as  $t \rightarrow \infty$ , and hence with probability 1 all components of CTMC  $\xi(t)$  explode simultaneously;

- 4) if  $\alpha > 0$  then CTMC  $\xi(t)$  is explosive. Moreover,

- i) if  $\beta < \alpha$  then with probability 1 the event (12) occurs and a single component of CTMC  $\xi(t)$  explodes;
- ii) if  $\alpha < \beta$  then with probability 1 the event  $B_{x,y_i}$  occurs for some  $i = 1, \dots, n$ , so that with probability 1 only a pair of adjacent components of the CTMC explodes.

### 3 Random walk in the quarter plane

Let graph  $\Lambda$  be formed by two adjacent vertices. This is the simplest example of both constant degree graphs and star graphs, while the corresponding Markov chain is equivalent to an inhomogeneous random walk on the positive quarter plane. We will briefly comment on this particular case to illustrate some distinctive features of the model dynamics, which can be also observed in more general situations.

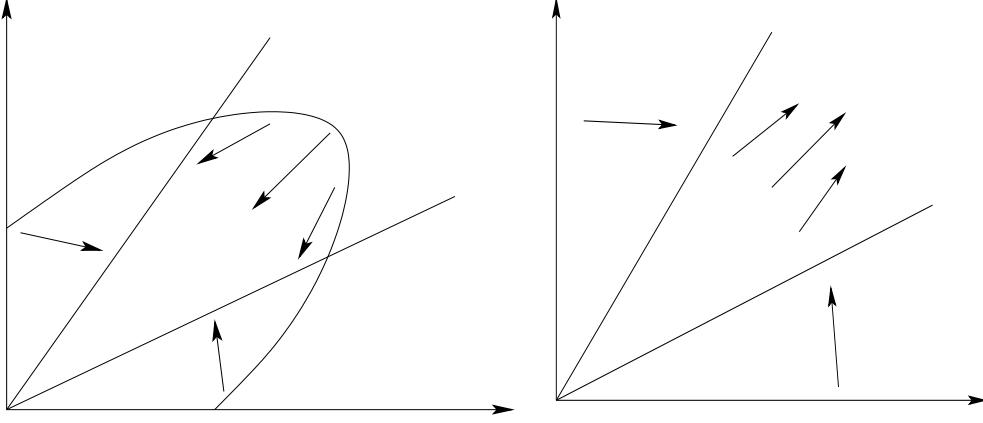


Figure 1: The vector field of mean jumps of Markov chains,  $\alpha < 0, \beta > 0$ . The vertical axis is  $y$  axis and the horizontal axis  $x$  axis. the upper line is  $y = -\frac{\alpha}{\beta}x$ , the lower line is  $y = -\frac{\beta}{\alpha}x$ , the curve is  $Q(x, y) = C$ , for some  $C > 0$ . Right:  $\alpha + \beta > 0$  (transience); the upper line is  $y = -\frac{\beta}{\alpha}x$ , the lower line is  $y = -\frac{\alpha}{\beta}x$ .

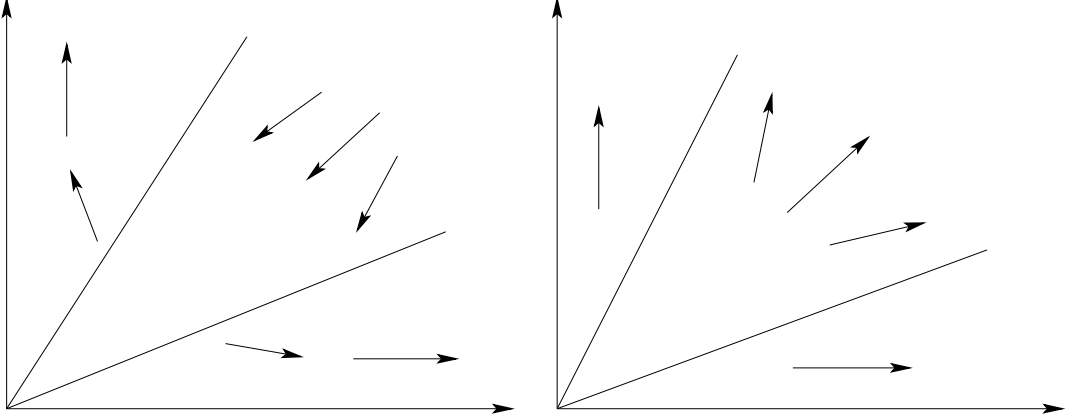


Figure 2: The vector field of mean jumps of Markov chains,  $\alpha > 0, \beta < 0$ . The vertical axis is  $y$  axis and the horizontal axis is  $x$  axis. the upper line is  $y = -\frac{\beta}{\alpha}x$ , the lower line is  $y = -\frac{\alpha}{\beta}x$ . Right:  $\alpha + \beta > 0$ ; the upper line is  $y = -\frac{\alpha}{\beta}x$ , the lower line is  $y = -\frac{\beta}{\alpha}x$ .

The theorems in Section 2 imply the following results for the two-dimensional case.

1) If  $\alpha < 0$  and  $\beta < |\alpha|$  then both CTMC  $\xi(t)$  and DTMC  $\zeta(t)$  are ergodic. Left part of Figure 1 sketches the vector field of mean jumps of the Markov chain and level curves of Lyapunov function  $Q(x, y) = -\alpha x^2 - \alpha y^2 - 2\beta xy$  in the ergodic case  $0 < \beta < -\alpha$ .

2) If  $\alpha < 0$  and  $\alpha + \beta \geq 0$  then DTMC  $\zeta(t)$  is transient, CTMC  $\xi(t)$  is explosive; moreover.

$$P(\zeta_1(t) = \zeta_2(t) \text{ infinitely often}) = 1.$$

The vector field of mean jumps in the case  $\alpha < 0, \alpha + \beta > 0$  is illustrated by the right part of Figure 1.

3) If  $\alpha > 0$  then CTMC  $\xi(t)$  is explosive. If, in addition,  $\beta < \alpha$  then with probability 1 a single component of DTMC will eventually grow (event (12) occurs). We illustrate this by the left part of Figure 2 in the case  $\beta < -\alpha < 0$ . The right part of the same figure corresponds to the transient/explosive case  $-\alpha < \beta < 0$ . If  $\alpha < \beta$  then both components grow and

$$P(\zeta_1(t) = \zeta_2(t) \text{ infinitely often}) = 1.$$

4) In the two-dimensional case we also deal with the case  $\alpha = 0$  and  $\beta < 0$  and show that both CTMC  $\xi(t)$  and DTMC  $\zeta(t)$  are null recurrent. Indeed, by the well-known criteria for recurrence (e.g., Theorem 2.2.1 in [4]) to show recurrence in both cases it suffices to find a positive function  $f(x, y)$  such that  $f(x, y) \rightarrow \infty$  as  $\sqrt{x^2 + y^2} \rightarrow \infty$  and for which the following inequality

$$Lf(x, y) \leq 0, \tag{14}$$

holds for all but finitely many  $(x, y)$ , where  $L$  is the generator of the Markov chain, see (4). Consider a function  $f(x, y) = \log(x + y + 1)$ . We will show that if the sum  $x + y$  is sufficiently large, then the inequality (14) holds. Indeed, if  $y = 0$  then

$$\begin{aligned} Lf(x, 0) &= (\log(x + 2) - \log(x + 1)) (1 + e^{\beta x}) + (\log(x) - \log(x + 1)) \\ &= \log \frac{x + 2}{x + 1} + \log \frac{x}{x + 1} + e^{\beta x} \log \frac{x + 2}{x + 1} \leq \log \left( 1 - \frac{1}{(x + 1)^2} \right) + e^{\beta x} \leq 0, \end{aligned}$$

where the last inequality holds for sufficiently large  $x > 0$ . If both  $x > 0$  and  $y > 0$  then assuming that  $C = x + y$  is large enough, we have the following bound:

$$\begin{aligned} Lf(x, y) &= (\log(C + 2) - \log(C + 1)) (e^{\beta y} + e^{\beta x}) \\ &\quad + (\log(C) - \log(C + 1)) (1_{\{x > 0\}} + 1_{\{y > 0\}}) \\ &\leq 2 (\log(C + 2) - 2 \log(C + 1) + \log(C)) = 2 \log \frac{C(C + 2)}{(C + 1)^2} \leq 0. \end{aligned}$$

Therefore, both Markov chains are recurrent. It is easy to see that CTMC  $\xi(t)$  cannot be positive recurrent in this case. Indeed, had it been recurrent, then its stationary distribution would be given by formula (7), but the latter is impossible, since

$$\sum_{x, y \in \mathbb{Z}_+} e^{\beta xy} = \infty$$

for all  $\beta$ . This also yields that DTMC  $\zeta(t)$  cannot be positive recurrent as well.



## 4 Proofs

### 4.1 Proof of Theorem 1

*Proof of Part 1) of Theorem 1.* Notice first that if  $\alpha < 0$  and  $\beta \leq 0$  then the stationary distribution (7) is well-defined and the Markov chain  $\xi(t)$  is ergodic.

We will show now that CTMC does not explode, if  $\alpha < 0$ ,  $\beta > 0$  and  $\alpha + \beta \max_{x \in \Lambda} \nu(x) \leq 0$ . Define

$$\tau_N = \min \left\{ t : \max_{x \in \Lambda} \xi_x(t) = N \right\}.$$

It is obvious that the Markov chain is explosive if and only if

$$\mathbb{P} \left( \lim_{N \rightarrow \infty} \tau_N < \infty \right) > 0,$$

but the latter cannot happen. Indeed, given  $\xi(t) = \xi$  let  $x \in \Lambda$  be such that  $\xi_x = \max_{y \in \Lambda} \xi_y$ . Then

$$U(x, \xi) = \alpha \xi_x + \beta \phi(x, \xi) \leq (\alpha + \beta \nu(x)) \xi_x \leq \left( \alpha + \beta \max_{x \in \Lambda} \nu(x) \right) \xi_x \leq 0.$$

Therefore the waiting times  $\tau_{N+1} - \tau_N$  are stochastically larger than exponentially distributed independent random variables with parameter  $(2|\Lambda|)^{-1}$ ; as a result, the limit  $\lim_{N \rightarrow \infty} \tau_N$  is infinite with probability 1 and thus the chain does not explode.

Let us finally show that if

$$\alpha < 0, \beta > 0, \alpha + \beta \max_{x \in \Lambda} \nu(x) < 0, \quad (15)$$

then  $Z_{\alpha, \beta, \Lambda} < \infty$  and consequently the stationary probability distribution is well defined. It is easy to see that  $Z_{\alpha, \beta, \Lambda} < \infty$  if and only if

$$\sum_{\xi \in \mathbb{Z}_+^\Lambda} \exp(-Q(\xi)/2) < \infty. \quad (16)$$

Consider a symmetric matrix  $A_Q = (a_{xy})_{x, y \in \Lambda}$  determining the quadratic form  $Q$ , i.e.

$$Q(u) = (A_Q u, u), \quad u \in \mathbb{R}^{|\Lambda|}. \quad (17)$$

It is easy to see that  $a_{xx} = -\alpha$ ,  $a_{xy} = -\beta$ , if  $y \sim x$  and  $a_{xy} = 0$  otherwise. Therefore for all  $x \in \Lambda$

$$|a_{xx}| - \sum_{y \neq x} |a_{xy}| = -\alpha - \beta \nu(x) \geq -\alpha - \beta \max_{x \in \Lambda} \nu(x) > 0,$$

because of (15). In other words, matrix  $A_Q$  is strictly diagonally dominant with positive diagonal entries and hence, by standard algebra, is positive definite. One can also observe positive definiteness of  $A_Q$  in the case under consideration from representation (11). Positive definiteness of  $A_Q$  implies that

$$\int_{\mathbb{R}^{|\Lambda|}} e^{-(A_Q u, u)/2} du = \frac{(2\pi)^{\frac{|\Lambda|}{2}}}{\sqrt{\det(A_Q)}} < \infty,$$

which, in turn, implies (16), so the stationary probability distribution is well defined as claimed.

*Proof of Part 2) of Theorem 1.* The Markov chain  $\xi(t)$  cannot be ergodic if  $\alpha \geq 0$ . Indeed, fix  $x \in \Lambda$  and define a set of configurations  $D_x = \{\xi : \xi_x \geq 0, \xi_y = 0, y \neq x\}$ . It is easy to see that

$$Z_{\alpha, \beta, \Lambda} \geq \sum_{\xi \in D_x} e^{W(\xi)} = \sum_{k=0}^{\infty} e^{\alpha k(k-1)/2} = \infty,$$

and the stationary distribution does not exist.

**Function  $Q$  as the Lyapunov function for Foster criteria.** Observe that ergodicity of the Markov chain in Part 1) of Theorem 1 can be shown by using Foster criteria for ergodicity of a countable Markov chain. We skip the easy case, when both  $\alpha < 0$  and  $\beta < 0$  and show that if  $\alpha < 0, \beta > 0$  and  $\alpha + \beta \max_{x \in \Lambda} \nu(x) < 0$  the function  $Q$  serves as the corresponding Lyapunov function. Indeed, the equation (11) yields that  $Q(\xi) > 0$  for all  $\xi \in \mathbb{Z}_+^\Lambda$  outside the origin (i.e.,  $\xi \neq 0$ ) and that  $Q(\xi) \rightarrow \infty$  as  $\sum_{x \in \Lambda} \xi_x^2 \rightarrow \infty$ . Recall that  $L$  is the generator (defined by (4)) of the Markov chain. We fix some  $\varepsilon > 0$  and show that

$$LQ(\xi) \leq -\varepsilon, \tag{18}$$

provided that  $S(\xi) = \sum_{x \in \Lambda} \xi_x \geq C$ , where  $C = C(\varepsilon)$  is sufficiently large. It is easy to see that

$$LQ(\xi) = \sum_{x \in \Lambda} (-\alpha - 2U(x, \xi))e^{U(x, \xi)} + \sum_{x \in \Lambda} (-\alpha + 2U(x, \xi))1_{\{\xi_x > 0\}}, \tag{19}$$

where  $U(x, \xi)$  is defined by equation (2). Sums in (19) can be respectively bounded as follows

$$\sum_{x \in \Lambda} (-\alpha - 2U(x, \xi))e^{U(x, \xi)} \leq |\Lambda| \max_{u \in \mathbb{R}} (-\alpha + 2u)e^{-u} = 2|\Lambda|e^{\frac{-\alpha-2}{2}},$$

and

$$\begin{aligned} \sum_{x \in \Lambda} (-\alpha + 2U(x, \xi))1_{\{\xi_x > 0\}} &\leq \sum_{x \in \Lambda} (-\alpha + 2U(x, \xi)) = -\alpha|\Lambda| + 2 \sum_{x \in \Lambda} (\alpha + \beta \nu(x))\xi_x \\ &\leq -\alpha|\Lambda| + 2(\alpha + \beta \max_{x \in \Lambda} \nu(x))S(\xi) \\ &\leq -\alpha|\Lambda| + 2C(\alpha + \beta \max_{x \in \Lambda} \nu(x)), \end{aligned}$$

where we used the equation (3) to get the second display. Thus the LHS of (18) is bounded by the following quantity

$$2|\Lambda|e^{\frac{-\alpha-2}{2}} - \alpha|\Lambda| + 2C(\alpha + \beta \max_{x \in \Lambda} \nu(x)),$$

which is less than  $-\varepsilon$ , if  $C > 0$  is sufficiently large. The inequality (18) allows to apply Foster criteria of ergodicity (Theorem 2.2.3 in [4]) of a countable Markov chain.

## 4.2 Proof of Theorem 2

We start with showing that there exists a  $\delta' > 0$  such that

$$\mathbf{P}(B|\zeta(t) = \zeta) > \delta', \quad (20)$$

for all  $t$  and  $\zeta$ . Given  $\zeta \in \mathbb{Z}_+^\Lambda$  define

$$M_\zeta = \max_{x \in \Lambda} U(x, \zeta) \quad \text{and} \quad D_\zeta = \{x \in \Lambda : U(x, \zeta) = M_\zeta\}.$$

Given  $\{\alpha, \beta\}$  there can be two different cases.

1) A finite connected graph  $\Lambda$  is such that

$$M_\zeta \geq 0 \text{ for all } \zeta \in \mathbb{Z}_+^\Lambda. \quad (21)$$

We say in this case that  $\Lambda$  is a type I graph.

2) The set of configurations

$$\mathcal{K} = \{\zeta : M_\zeta < 0\}, \quad (22)$$

is not empty, then we say that  $\Lambda$  is a type II graph.

Let us consider some examples before proceeding further. It is obvious that if both  $\alpha$  and  $\beta$  are positive, then any graph is a type I graph. Also, if  $\alpha > 0 > \beta$  and  $\alpha + \beta \max_{x \in \Lambda} \nu(x) \geq 0$  then for every  $x \in \Lambda$  such that  $\zeta_x = N = \max_{y \in \Lambda} \zeta_y$  the following inequality holds

$$U(x, \zeta) = \alpha N + \beta \phi(x, \zeta) \geq N \left( \alpha + \beta \max_{x \in \Lambda} \nu(x) \right) \geq 0,$$

hence,  $\Lambda$  is a type I graph.

Consider also two main examples of type II graphs. The first one is when  $\alpha > 0 > \alpha + \beta\nu$ , and  $\Lambda$  is a constant vertex degree graph with  $\nu(x) \equiv \nu$ . In this case  $\mathcal{K}$  is a non-empty set of configurations that belongs to intersection of hyperplanes  $\{\zeta \in \mathbb{R}_+^\Lambda : \alpha\zeta_x + \beta\phi(x, \zeta) < 0, x \in \Lambda\}$ . The second example is when  $\alpha > 0 > \alpha + \beta\sqrt{\nu}$  and  $\Lambda$  is a star graph with  $n+1$  vertices. In this case the set  $\mathcal{K}$  is the subset of intersection of hyperplanes  $\{\zeta \in \mathbb{R}_+^\Lambda : \alpha\zeta_x + \beta\phi(x, \zeta) < 0, x \in \Lambda\}$ . Notice also that if set  $\mathcal{K}$  not empty, then it is infinite. Indeed, if  $\zeta \in \mathcal{K}$  then  $a\zeta \in \mathcal{K}$  for any  $a \in \mathbb{Z}_+$ .

Let us continue with the proof of (20). First, given a pair  $\{\alpha, \beta\}$  suppose that  $\Lambda$  is a type I graph. For a given  $x \in \Lambda$  define the following event

$$B_{t,x} = \{\zeta_x(s+1) = \zeta_x(s) + 1, \zeta_y(s) = \zeta_y(t), \text{ for } y \neq x \text{ and } s \geq t\}.$$

Trivially,  $B_{t,x} \subset B$ . We are going to show that for any  $\zeta$  and  $x \in D_\zeta$

$$\mathbf{P}(B_{t,x} | \zeta(t) = \zeta) > \delta' > 0,$$

where  $\delta'$  might depend only on parameters  $\alpha, \beta$  and graph  $\Lambda$ . Given  $x \in \Lambda$  and  $\zeta \in \mathbb{Z}_+^\Lambda$  denote

$$R(x, \zeta) = \sum_{y \in \Lambda} e^{U(y, \zeta)} - \left( e^{U(x, \zeta)} + \sum_{y \sim x} e^{U(y, \zeta)} \right).$$

If  $x \in D_\zeta$  then

$$R(x, \zeta)e^{-U(x, \zeta)} = R(x, \zeta)e^{-M_\zeta} \leq (|\Lambda| - \nu(x) - 1) < |\Lambda|, \quad (23)$$

for all  $\zeta \in \mathbb{Z}_+^\Lambda$ . Given  $x \in D_\zeta$  we have that

$$\begin{aligned} \mathbb{P}(B_{t,x} | \zeta) &= \prod_{k=0}^{\infty} \frac{e^{M_\zeta + \alpha k}}{e^{M_\zeta + \alpha k} + \sum_{y \sim x} e^{U(y, \zeta) + \beta k} + R(x, \zeta) + \sum_{y \in \Lambda} 1_{\{\zeta_y > 0\}}} \\ &= \prod_{k=0}^{\infty} \frac{1}{1 + \sum_{y \sim x} e^{U(y, \zeta) - M_\zeta - (\alpha - \beta)k} + \left[ R(x, \zeta) + \sum_{y \in \Lambda} 1_{\{\zeta_y > 0\}} \right] e^{-M_\zeta - \alpha k}}, \end{aligned}$$

for all  $\zeta \in \mathbb{Z}_+^\Lambda$ . It is easy to see that by choice of  $x$  we have

$$\sum_{y \sim x} e^{U(y, \zeta) - M_\zeta - (\alpha - \beta)k} \leq e^{-(\alpha - \beta)k} \max_{y \in \Lambda} \nu(y).$$

Also, using (21) and (23) we get that

$$\left( R(x, \zeta) + \sum_{y \in \Lambda} 1_{\{\zeta_y > 0\}} \right) e^{-M_\zeta - \alpha k} \leq 2|\Lambda|e^{-\alpha k}. \quad (24)$$

Therefore, we obtain the following bound

$$\mathbb{P}(B_{t,x} | \zeta) \geq \prod_{k=0}^{\infty} \frac{1}{1 + e^{-(\alpha - \beta)k} \max_{y \in \Lambda} \nu(y) + 2|\Lambda|e^{-\alpha k}} = \delta' > 0. \quad (25)$$

The preceding display implies bound (20) in the case of a type I graph.

Suppose now that given a pair  $\alpha, \beta$  satisfying the conditions of the theorem,  $\Lambda$  is a type II graph. Fix some  $\varepsilon > 0$  and suppose that  $\zeta \in \mathcal{K}_\varepsilon = \{\zeta : M_\zeta \geq -\varepsilon\}$ . Given  $x \in D_\zeta$  one can repeat, with a minor change, the same argument which led to bound (25). The only difference now is that the inequality  $M_\zeta \geq -\varepsilon$  yields constant  $(1 + e^\varepsilon)|\Lambda|e^{-\alpha k}$  in the right side of (24) (instead of  $2|\Lambda|e^{-\alpha k}$ ) and it results in a different  $\delta'' \neq \delta'$  such that

$$\mathbb{P}(B_{t,x} | \zeta(t) = \zeta) > \delta'' > 0.$$

Consider the case, when  $\zeta \in \mathcal{K}_\varepsilon^c = \{\zeta : M_\zeta < -\varepsilon\}$ . Define a stopping time

$$\tau = \min\{t : \zeta(t) \in \mathcal{K}_\varepsilon\}.$$

It is easy to see that  $\mathbb{P}(\tau < \infty | \zeta) = 1$  for all  $\zeta \in \mathcal{K}_\varepsilon^c$ . Indeed, define  $F(\zeta) = |\zeta|^2$ , where  $|\zeta|$  is Euclidean norm in  $\mathbb{R}^\Lambda$ . A direct computation gives that there exists some  $\varepsilon' > 0$  such that

$$\mathbb{E}(F(\zeta(t+1)) - F(\zeta(t)) | \zeta(t) = \zeta) \leq -\varepsilon'$$

for all  $\zeta \in \mathcal{K}_\varepsilon^c$ . The inequality in the preceding display yields, by a standard argument (see, for example, Theorem 2.1.1 in [4]), that starting from inside of  $\mathcal{K}_\varepsilon^c$  the process  $\zeta(t)$  reaches the

set  $\{\zeta : |\zeta| \leq C\} \cap \mathcal{K}_\varepsilon^c$  in a finite time unless it first exits the set  $\mathcal{K}_\varepsilon^c$ . Suppose the process reaches set  $\{\zeta : |\zeta| \leq C\}$  before exiting  $\mathcal{K}_\varepsilon^c$ . DTMC  $\zeta(t)$  transits to  $\mathcal{K}_\varepsilon$  from  $\{\zeta : |\zeta| \leq C\}$  with positive probability due to irreducibility of DTMC and finiteness of set  $\{\zeta : |\zeta| \leq C\}$ . This implies that DTMC  $\zeta(t)$  reaches set  $\mathcal{K}_\varepsilon$  with probability 1 and we can apply the same argument as in the case of type I graph to obtain the following bound

$$\mathbb{P}(B | \zeta(t) = \zeta) = \mathbb{P}(B, \tau < \infty | \zeta) \geq \min_{\zeta \in \mathcal{K}_\varepsilon^c, x \in D_\zeta} \mathbb{P}(B_{\tau, x} | \zeta) > \delta'' > 0.$$

for all  $\zeta \in \mathcal{K}_\varepsilon^c$ . This completes the proof of bound (20).

Bound (20) implies that

$$\mathbb{E}(1_B | \mathcal{F}_t) > \delta', \quad (26)$$

where  $\mathcal{F}_t = \sigma\{\zeta_0, \dots, \zeta_t\}$  is the  $\sigma$ -algebra of events generated by DTMC up to time moment  $t$ . Since  $B \in \mathcal{F}_\infty = \sigma\{\mathcal{F}_t, t \geq 0\}$  we get by Levi 0 – 1 law that

$$\mathbb{E}(1_B | \mathcal{F}_t) \rightarrow \mathbb{E}(1_B | \mathcal{F}_\infty) = 1_B, \text{ as } t \rightarrow \infty.$$

By (26) the right hand side of the preceding display is positive. Therefore, it must be equal to 1, hence,  $\mathbb{P}(B) = 1$ .

Thus, eventually only a single component of the embedded chain continues to evolve by jumping up without jumping down. In the continuous time setting the only growing component evolves eventually as a pure birth process with exponentially growing birth rates. The latter process is explosive and, hence, CTMC  $\xi(t)$  is explosive, where with probability 1 only a single component explodes.

Notice also that under the assumptions of the theorem with probability one a typical trajectory of DTMC  $\zeta(t)$  returns to set  $\mathcal{K}_\varepsilon^c$  only a finite number of times in the case of type II graph.

### 4.3 Proof of Theorem 3

We start with the following lemma.

**Lemma 1** *Let  $0 < \alpha < \beta$ . Suppose that  $x_1$  and  $x_2$  are two vertices of  $\Lambda$  such that (1)  $x_1 \sim x_2$ ; (2) there is no  $y$  such that  $y \sim x_1$  and  $y \sim x_2$  at the same time; (3) at some time  $s$  the configuration of the DTMC is such that  $u_1 = U(x_1, \zeta(s))$  is the largest potential on the whole graph and  $u_2 = U(x_2, \zeta(s))$  is the largest potential among all the neighbours of  $x_1$ . Then, with a positive probability depending on  $\alpha$ ,  $\beta$  and  $\Lambda$  only, the following events simultaneously occur*

$$\begin{aligned} & \zeta_y(t) = \zeta_y(s) \text{ for all } y \notin \{x_1, x_2\} \text{ and all } t = s, s+1, s+2, \dots; \\ & \lim_{t \rightarrow \infty} \frac{\zeta_{x_1}(t)}{t} = \lim_{t \rightarrow \infty} \frac{\zeta_{x_2}(t)}{t} = \frac{1}{2}. \end{aligned}$$

*Proof of Lemma 1.* Observe that every time when the component at  $x_1$  increases by 1, the potential at  $x_1$  increases by  $\alpha$  while the potential at each of the neighbours of  $x_1$  increases by  $\beta$ , therefore the potential at  $x_2$  remains the largest among the neighbours of  $x_1$ . At the same time the difference between the potentials at  $x_1$  and  $x_2$  decreases by  $\delta := \beta - \alpha > 0$ .

Let  $k = \lfloor \frac{u_1 - u_2}{\delta} \rfloor$  where  $\lfloor a \rfloor$  denotes the integer part of  $a \in \mathbb{R}$ . W.l.o.g. assume that  $k$  is even. Denoting by  $\nu_1 = \nu(x_1)$  the degree of vertex  $x_1$  (and  $\nu_2 = \nu(x_2)$  respectively), we obtain that the probability that during the times  $t = s, s+1, \dots, s+k$  only the component at  $x_1$  increases is larger than

$$\begin{aligned}
p_1 &= \prod_{i=0}^k \frac{e^{u_1+i\alpha}}{e^{u_1+i\alpha} + \nu_1 e^{u_2+i\beta} + (|\Lambda| - \nu_1)e^{u_1}} \\
&= \prod_{i=0}^k \frac{1}{1 + \nu_1 e^{-[u_1-u_2]+i(\beta-\alpha)} + (|\Lambda| - \nu_1)e^{-i\alpha}} \geq \prod_{i=0}^k \frac{1}{1 + \nu_1 e^{-(k-i)\delta} + |\Lambda|e^{-i\alpha}} \\
&\geq \prod_{i=0}^{k/2} \frac{1}{1 + \nu_1 e^{-k\delta/2} + |\Lambda|e^{-i\alpha}} \times \prod_{j=0}^{k/2} \frac{1}{1 + \nu_1 e^{-j\delta} + |\Lambda|e^{-k\alpha/2}} \\
&\geq \left( \prod_{i=0}^{k/2} \frac{1}{1 + (\nu_1 + |\Lambda|)(e^{-i\delta} + e^{-i\alpha})} \right)^2 = C_1(|\Lambda|, \alpha, \beta) > 0.
\end{aligned}$$

Consequently, by time  $s+k$  we have  $-\delta < U(x_2, \zeta(s+k)) - U(x_1, \zeta(s+k)) \leq 0$  with probability at least  $p_1$ .

From now on assume w.l.o.g. that actually  $u_2 \in (u_1 - \delta, u_1]$  already at time  $s$ . Let  $m_i(t)$ ,  $i = 1, 2$  be the number of times  $x_i$  was chosen during the times  $s+1, s+2, \dots, s+t$ . Define the events

$$\begin{aligned}
A'_k &= \{\zeta_y(s+i) = \zeta_y(s) \text{ for all } y \notin \{x_1, x_2\}, i = 1, 2, \dots, 2k^2\} \\
A''_k &= \{|m_1(2k^2) - m_2(2k^2)| \leq 2k\} \\
A_k &= A'_k \cap A''_k.
\end{aligned}$$

Then under  $A_k$  we have  $m_1(2k^2) + m_2(2k^2) = 2k^2$  and  $|m_i(2k^2) - k^2| \leq k$  for  $i = 1, 2$ , so  $P(A'_{k+1} | A_k)$  is no less than

$$\begin{aligned}
&\prod_{i=0}^{4k+1} \frac{e^{U(x_1, \zeta(s+2k^2+i))} + e^{U(x_2, \zeta(s+2k^2+i))}}{e^{U(x_1, \zeta(s+2k^2+i))} + e^{U(x_2, \zeta(s+2k^2+i))} + [\nu_1 + \nu_2]e^{u_2+\beta(k^2+6k)} + [|\Lambda| - \nu_1 - \nu_2]e^{u_2}} \\
&\geq \prod_{i=0}^{4k+1} \frac{1}{1 + |\Lambda|e^{(7\beta+\alpha)k-\alpha k^2}} \geq 1 - C_2(|\Lambda|, \alpha)e^{-k}
\end{aligned}$$

since  $U(x_1, \zeta(s+2k^2+i)) \geq u_1 + \alpha(k^2 - k) + \beta(k^2 - k)$ , and the potential at any  $y \sim x_1$  or  $\sim x_2$  is bounded by  $u_2 + \beta(k^2 + k + (4k+1)) \leq u_2 + \beta(k^2 + 6k)$ . To estimate  $P(A''_{k+1} | A_k)$  observe that whenever  $m_1(j) > m_2(j) + 1$  the potential at  $x_2$  is larger, and the similar statement holds if one swaps 1 and 2. Now, there are two possibilities at time  $j = s + 2k^2$ : (a)  $|m_1(2k^2) - m_2(2k^2)| \leq 1.5k$  and (b)  $|m_1(2k^2) - m_2(2k^2)| > 1.5k$ .

In case (a), the difference  $|m_1(j) - m_2(j)|$  can be majorized by the distance to the origin of the simple symmetric random walk on  $\mathbb{Z}^1$ . In particular, the probability that during  $4k+2$  steps it is further than  $k^{2/3}$  from the starting point is bounded by  $c_3 e^{-k^{1/6}}$  where  $c_3$  is some constant.

As a result, with probability at least  $1 - c_3 e^{-k^{1/6}}$  we have  $|m_1(2(k+1)^2) - m_2(2(k+1)^2)| < 1.5k + k^{2/3} < 2(k+1)$  and  $A''_{k+1}$  occurs.

On the other hand, in case (b) we have  $1.5k < |m_1(2k^2) - m_2(2k^2)| \leq 2k$ , hence the potential at the larger  $x_i$  in the pair  $\{x_1, x_2\}$  is much smaller than the potential at the smaller  $x$  in this pair. Consequently, for the next  $k$  steps the probability to increase the larger component, divided by the probability to increase the smaller component, is bounded above by  $e^{-\delta k/2}$ , so we can couple  $|m_1(j) - m_2(j)|$  with an asymmetric simple random walk on  $\mathbb{Z}^1$  with the drift towards the origin. As a result, we obtain that with probability at least  $1 - e^{-c_4 k}$  during the times  $t = s + 2k^2 + i$ ,  $i = 1, \dots, k$ , the distance between  $m_1$  and  $m_2$  decreases at least by  $k/2$ , bringing it to the value less than  $2k - (k/2) = 1.5k$ , and thus to case (a). Therefore,

$$P(A''_{k+1} | A_k) \geq 1 - C_3 e^{-k^{1/6}} - e^{-C_4 k}.$$

Combining the above inequalities yields

$$P(A_{k+1} | A_k) \geq 1 - C_3 e^{-k^{1/6}} - e^{-C_4 k} - C_2(|\Lambda|, \alpha) e^{-k}. \quad (27)$$

Since the product of the terms on the RHS of (27) over all large enough  $k$  is positive, the statement of the lemma follows.

Now note that at any moment of time  $s$  there is a vertex  $x_1$  with the largest potential. Because of our assumption it satisfies the conditions of Lemma 1 for *some* neighbour  $x_2$ . Hence, Theorem 3 follows from the Levy 0–1 law.

## 4.4 Proof of Theorem 4

*Proof of Part 1) of Theorem 4.* Non-ergodicity in the case  $\alpha \geq 0$  and ergodicity in the case  $\alpha < 0$ ,  $\alpha + \beta\nu < 0$  are implied by Theorem 1. If  $\alpha < 0$ ,  $\alpha + \beta\nu \geq 0$  then, using equations (10) and (11), we get the the following bound

$$Z_{\alpha, \beta, \Lambda} \geq \sum_{\xi \in Z_+^\Lambda} e^{W(\xi)} 1_{\{\xi: \xi_x = \xi_y, \forall x, y \in \Lambda\}} = \sum_{k=1}^{\infty} e^{|\Lambda|((\alpha + \beta\nu)k^2 - \alpha k)/2} = \infty,$$

which means that the stationary distribution does not exist in this case and, hence, the CMTC is not ergodic.

Notice, in addition, that if  $\Lambda$  is a constant vertex degree graph then  $(-\alpha - \beta\nu)$  is the eigenvalue of  $A_Q$  with the corresponding eigenvector  $(1, \dots, 1)$  and, hence, the function  $\exp(-Q(\xi)/2)$  is not summable in the direction of this eigenvector, provided that  $-\alpha - \beta\nu \geq 0$ . Furthermore, if  $\alpha < 0$ ,  $\beta > 0$  then  $-\alpha - \beta\nu$  is the minimal eigenvalue of  $A_Q$ , since all eigenvalues of matrix  $A_Q$  lie, by Gershgorin circle theorem, within the closed interval  $[-\alpha - |\beta|\nu, -\alpha + |\beta|\nu]$ .

Further, it is rather straightforward to compute the characteristic polynomial of matrix  $A_Q$  in the case of the mean-field model with  $n$  vertices (complete graph with  $n$  vertices). This polynomial is

$$(-1)^{n-1}(\alpha - \beta + \mu)^{n-1}(-\alpha - (n-1)\beta - \mu),$$

and analysis of the eigenvalues yields the same results for a complete graph.

## Proofs for non-ergodic cases.

**Lemma 2** *Let  $\Lambda$  be a finite connected graph with the constant vertex degree  $\nu(x) \equiv \nu$ . If  $\alpha + \beta\nu > 0$  then with probability 1 there exists a time moment  $\tau < \infty$  such that for all  $t \geq \tau$  none of the components of DTMC  $\zeta(t)$  decreases.*

*Proof of Lemma 2.* Recall that  $U(x, \zeta)$  is the quantity defined by equation (2) and the quantity  $S(\zeta)$  is defined by (9). Since  $\alpha + \beta\nu > 0$  equation (3) implies that for all  $\zeta$

$$\max_{x \in \Lambda} U(x, \zeta) \geq C_1 S(\zeta), \quad (28)$$

where  $C_1 = (\alpha + \beta\nu)/|\Lambda|$ . Using this bound for the maximal potential we get the following inequality

$$\begin{aligned} P(S(\zeta(t+1)) = S(\zeta(t)) + 1 \mid \zeta(t) = \zeta) &= 1 - \frac{\sum_{x \in \Lambda} 1_{\{\zeta_x > 0\}}}{\sum_{x \in \Lambda} e^{U(x, \zeta)} + \sum_{x \in \Lambda} 1_{\{\zeta_x > 0\}}} \\ &\geq 1 - \frac{|\Lambda|}{\max_{x \in \Lambda} e^{U(x, \zeta)}} \geq 1 - |\Lambda| e^{-C_1 S(\zeta)}. \end{aligned}$$

Therefore, if  $D_s = \{\text{none of the components ever decreases after time } s\}$ , then

$$P(D_s \mid \zeta(s) = \zeta) \geq \prod_{t=s}^{\infty} (1 - C_2(\zeta) e^{-C_1(t-s)}) = 1 - o(S(\zeta)) \quad (29)$$

where  $C_2(\zeta) = |\Lambda| e^{-C_1 S(\zeta)}$  and  $o(S(\zeta)) \rightarrow 0$  as  $S(\zeta) \rightarrow \infty$ . Since for any  $N > 0$  the set of configurations  $\{\zeta : S(\zeta) \geq N\}$  is finite and the Markov chain is irreducible, for each  $N = 1, 2, \dots$ , we can define  $\tau_N = \min\{t : S(\zeta(t)) = N\} < \infty$ . As  $P(D_{\tau_N}) \rightarrow 1$ , by continuity of probability  $P(\cup_N D_{\tau_N}) = 1$  and hence there exists  $N$  such that after time  $\tau = \tau_N$  the only changes in the system are increases of the components. This finishes the proof of Lemma 2.

It is obvious that Lemma 2 implies transience of the DTMC in the case  $\alpha + \beta\nu > 0$ . Nevertheless we would like to provide another lemma (Lemma 3 below) that ensures transience in this case. This lemma takes into account the geometry of mean jumps and formalizes intuition which can be inferred from, for example, right images in Figures 1 and 2. Besides, it provides an idea for proving transience in the case  $\alpha + \beta\nu = 0$  (see Lemma 4 below).

**Lemma 3** *Let  $\Lambda$  be a finite connected graph with the constant vertex degree  $\nu(x) \equiv \nu$ . If  $\alpha + \beta\nu > 0$ , then for any  $0 < \varepsilon < 1$  the following bound holds*

$$E(S(\zeta(t+1)) - S(\zeta(t)) \mid \zeta(t) = \zeta) \geq \varepsilon, \quad (30)$$

*provided that  $S(\zeta) > C_1 = \frac{2|\Lambda|\varepsilon}{(1-\varepsilon)(\alpha+\beta\nu)}$ .*

*Proof of Lemma 3.* It is easy to see that inequality (30) is equivalent to the following one

$$J(\zeta, \varepsilon) := \sum_{x \in \Lambda} (\delta(\varepsilon) e^{U(x, \zeta)} - 1_{\{\zeta_x > 0\}}) \geq 0, \quad (31)$$



where

$$\delta(\varepsilon) = \frac{1 - \varepsilon}{1 + \varepsilon}. \quad (32)$$

Using subsequently inequality  $1_{\{\zeta_x > 0\}} \leq 1$ , equation (3) and inequality  $e^u > 1 + u$ ,  $u \in \mathbb{R}$ , we obtain

$$\begin{aligned} J(\zeta, \varepsilon) &\geq \sum_{x \in \Lambda} (\delta(\varepsilon) e^{U(x, \zeta)} - 1) \\ &\geq \sum_{x \in \Lambda} (\delta(\varepsilon)(1 + U(x, \zeta)) - 1) \\ &= \delta(\varepsilon)(\alpha + \beta\nu)S(\zeta) - (1 + \delta(\varepsilon))|\Lambda| > 0, \end{aligned}$$

provided that  $S(\zeta) > C_1 = \frac{(1 + \delta(\varepsilon))|\Lambda|}{\delta(\varepsilon)(\alpha + \beta\nu)}$ . Notice that it is also possible to use inequality between the arithmetical and geometric means and equation (3) in order to obtain that

$$\sum_{x \in \Lambda} e^{U(x, \zeta)} \geq |\Lambda| e^{\frac{(\alpha + \beta\nu)S(\zeta)}{|\Lambda|}}$$

and to arrive to a similar result (provided that  $S(\zeta) > C_2$ , where  $C_2$  is another constant). Lemma 3 is proved.

Lemma 3 means that conditions of Theorem 2.2.7 in [4] are satisfied with the linear function  $f(\zeta) = S(\zeta)$  and set  $A = \{\zeta \in \mathbb{Z}_+^\Lambda : S(\zeta) \geq C_1\}$  and the embedded Markov chain  $\zeta(t)$  is transient in the case  $\alpha + \beta\nu > 0$ .

**Lemma 4** *Let  $\Lambda$  be a finite connected graph with the constant vertex degree  $\nu(x) \equiv \nu$ . If  $\alpha + \beta\nu = 0$ , then there exist  $\varepsilon > 0$  and  $C > 0$  such that the following bound holds*

$$\mathbb{E}(S(\zeta(t + k(\zeta(t)))) - S(\zeta(t)) \mid \zeta(t) = \zeta) \geq \varepsilon, \quad (33)$$

provided that  $S(\zeta) \geq C$  and where

$$k(\zeta) = \begin{cases} 1, & U(x, \zeta) \neq 0 \text{ for some } x \in \Lambda, \\ 2, & U(x, \zeta) = 0 \text{ for all } x \in \Lambda. \end{cases}$$

*Proof of Lemma 4.* As we already noted in the proof of Lemma 3 inequality (33) is equivalent to the following one

$$J(\zeta, \varepsilon) = \delta(\varepsilon) \sum_{x \in \Lambda} e^{U(x, \zeta)} - \sum_{x \in \Lambda} 1_{\{\zeta_x > 0\}} \geq 0, \quad (34)$$

where  $\delta(\varepsilon)$  is defined by (32) and (34) would be implied by

$$\delta(\varepsilon) \sum_{x \in \Lambda} e^{U(x, \zeta)} - |\Lambda| \geq 0.$$

Notice that by inequality between geometric and arithmetic means we have that for all  $\zeta$

$$\sum_{x \in \Lambda} e^{U(x, \zeta)} - |\Lambda| \geq 0, \quad (35)$$

since by equation (3)

$$\sum_{x \in \Lambda} U(x, \zeta) = (\alpha + \beta\nu)S(\zeta) = 0. \quad (36)$$

It is well known that given numbers  $a_1, \dots, a_m$  geometric and arithmetic means of these numbers are equal to each other if and only if  $a_1 = \dots = a_m$ . Therefore, equation (36) also implies that identity  $\sum_{x \in \Lambda} e^{U(x, \zeta)} - |\Lambda| = 0$  holds if and only if  $U(x, \zeta) = 0$ , for all  $x \in \Lambda$  otherwise we have got a strict inequality in (35). Thus, if there are exactly  $0 < m \leq |\Lambda|$  vertices with non zero potentials then

$$\sum_{x \in \Lambda} e^{U(x, \zeta)} - |\Lambda| \geq \sum_{x \in \Lambda: U(x, \zeta) \neq 0} e^{U(x, \zeta)} - m > 0.$$

It is easy to see that since the inequality in the preceding display is strict there exists  $\delta_m \in (0, 1)$  such that

$$\delta_m \sum_{x \in \Lambda} e^{U(x, \zeta)} - |\Lambda| \geq \delta_m \sum_{x \in \Lambda: U(x, \zeta) \neq 0} e^{U(x, \zeta)} - m > 0,$$

because values of potentials  $U$  belong to a discrete set  $\{\alpha(k - j/\nu), k, j \in \mathbb{Z}_+\}$  (where we used that  $\beta = -\alpha/\nu$ ) which is bounded away from zero. Thus, given  $0 < m \leq |\Lambda|$  we claim existence of  $\delta_m$  and, hence, existence of the corresponding  $\varepsilon = \varepsilon(\delta_m)$  (using equation (32)). The required in Lemma 4  $\varepsilon$  is obtained as  $\varepsilon = \min_m \varepsilon_m$ .

It is easy to see that all potentials cannot stay zero for two steps in a row, hence

$$\mathbb{E}(S(\zeta(t+2)) - S(\zeta(t)) | \zeta(t) = \zeta) = \mathbb{E}(S(\zeta(t+2)) - S(\zeta(t+1)) | \zeta(t) = \zeta) \geq \varepsilon.$$

Thus inequality (33) is proven, and by Theorem 2.2.7 in [4] the embedded Markov chain is transient. This completes the proof of Lemma 4.

We are ready now to finish the proof of Theorem 4.

*Proof of Part 2) of Theorem 4.* If  $\alpha + \beta\nu = 0$  then transience of DTMC  $\zeta(t)$  implies at least transience of CTMC  $\xi(t)$ . By Theorem 1 CTMC  $\xi(t)$  does not explode if  $\alpha < 0$ ,  $\alpha + \beta\nu = 0$ . Hence, CTMC  $\xi(t)$  is transient if  $\alpha < 0$ ,  $\alpha + \beta\nu = 0$ .

**Remark 3** Let us notice how the sign of parameter  $\alpha$  influences the process dynamics in the case  $\alpha + \beta\nu = 0$ . If  $\alpha > 0$ ,  $\alpha + \beta\nu = 0$ , then Theorem 2 applies (since  $\beta < 0$ ) and, eventually, a single component of the Markov chain explodes. A set of configurations  $\{\xi : \xi_x = \xi_y, x, y \in \Lambda\}$  is "unstable" in the sense that the process tends to leave it and to never return. In contrast, if  $\alpha < 0$ , then the process tends to stay in a neighbourhood of the same set of configurations (with equal components) while escaping to infinity. It is easy to see that vertex potentials are bounded around this set of configurations and this is why no explosion happens in this case.

*Proof of Part 3) of Theorem 4.* If  $\alpha + \beta\nu > 0$ , then explosiveness of CTMC  $\xi(t)$  is implied (regardless of the sign of  $\alpha$ ) by Lemma 2. Indeed, given a configuration  $\xi$  bound (28) implies the following lower bound for the total transition rate

$$\sum_{x \in \Lambda} (e^{U(x, \xi)} + 1_{\{\xi_x > 0\}}) \geq \max_{x \in \Lambda} e^{U(x, \xi)} \geq e^{C_1 S(\xi)},$$

where, as before,  $C_1 = (\alpha + \beta\nu)/|\Lambda|$ . Besides, none of the components decrease after time  $\tau$  defined in Lemma 2. Therefore the only changes in the systems are jumps up and these jumps happen with exponentially increasing rates whose inverses are summable. This yields explosion.

*Proof of Part 4) of Theorem 4.* If both  $\alpha > 0$  and  $\beta > 0$  then transience of DTMC  $\zeta(t)$  and explosiveness of CTMC  $\xi(t)$  are obvious. On the other hand, if  $\alpha > \max\{0, \beta\}$  then Theorem 2 applies; if  $0 < \alpha < \beta$  and the graph  $\Lambda$  is without triangles then Theorem 3 applies. The proof of Theorem 4 is finished.

## 4.5 Proof of Theorem 5

Let  $\zeta(t) = (\zeta_1(t), \dots, \zeta_n(t))$  be DTMC corresponding to a complete graph with  $n$  vertices. It is easy to see that the potential of a component at vertex  $i$  at time  $t$  is equal to

$$U(i, \zeta(t)) = \alpha\zeta_i(t) + \beta(S(\zeta(t)) - \zeta_i(t)) = (\alpha - \beta)\zeta_i(t) + \beta S(\zeta(t)).$$

First, we present an intuitive argument justifying the theorem, which is made rigorous later. In both cases described in the theorem,  $\alpha + \beta\nu > 0$  hence by Lemma 2 there exists a moment of time  $\tau$  after which none of the components decrease. Also, it is easy to see that in both cases of the theorem  $\beta$  must be positive. So, for  $t > \tau$  the probability that it is the  $i$ -th component that increases is equal to

$$\frac{e^{(\alpha-\beta)\zeta_i(t)+\beta S(\zeta(t))}}{\sum_{k=1}^n e^{(\alpha-\beta)\zeta_k(t)+\beta S(\zeta(t))}} = \frac{e^{(\alpha-\beta)\zeta_i(t)}}{\sum_{k=1}^n e^{(\alpha-\beta)\zeta_k(t)}}$$

Therefore, in the long run DTMC evolves as a generalized Pólya urn model with weight function  $g(x) = e^{(\alpha-\beta)x}$ . Now the well-known results for a generalized Pólya urn scheme and Theorem 1 in [12] implies Parts 1) and 3) of Theorem 5. Finally, the explosiveness of the process  $\xi(t)$  follows from Parts 3) and 4) of Theorem 4. (One can compare this and the following calculations with the argument presented in the proof of Part 3) of Theorem 6.)

The problem with the above argument is that, strictly speaking, the events  $\zeta_{i+1}(t) = \zeta_i(t) + 1$ ,  $i = 1, 2, \dots, n$ , and  $\tau < t$  are *not* independent, as the behaviour of the Pólya urn *may* affect the probability of decreasing of a component. Thus, to make the argument rigorous, we construct the following coupling.

Let  $Y_t$ ,  $t = 1, 2, \dots$ , be a sequence of i.i.d. uniform  $[0, 1]$  random variables. At time  $t$  split the interval  $[0, 1]$  into  $2n$  intervals with lengths proportional to

$$[e^{U(1, \zeta(t))}, e^{U(2, \zeta(t))}, \dots, e^{U(n, \zeta(t))}, 1, 1, \dots, 1]$$

where  $U$  is defined by (2). If  $Y_t$  falls into the  $i$ -th subinterval with  $1 \leq i \leq n$  then we set  $\zeta_i(t+1) = \zeta_i(t) + 1$ ; if  $n+1 \leq i \leq 2n$  then we set  $\zeta_i(t+1) = \max\{0, \zeta_i(t) - 1\}$ . In both cases we leave the remaining components unchanged. It is easy to see that the process  $\zeta(t)$ ,  $t \geq 1$ , has exactly the same distribution as the DTMC defined above. At the same time for a fixed  $N \in \mathbb{Z}^+$  define the process  $\zeta^{(N)}(t)$ ,  $t = N, N+1, \dots$ , such that  $\zeta^{(N)}(N) := \zeta(N)$  and the transition rules of  $\zeta^{(N)}(t)$  are exactly the same as that of  $\zeta(t)$  with the only exception that when  $Y_t$  falls in the interval with index  $\geq n+1$  the process  $\zeta^{(N)}(t)$  remains unchanged (i.e.,

“no deaths”). Let  $B_N$  be the event “none of  $Y_t$  falls in the intervals indexed  $n+1, n+2, \dots, 2n$  for all  $t \geq N$ ”, then on  $B_N$  we have  $\zeta^{(N)}(t) \equiv \zeta(t)$ ,  $t \geq N$ , consequently  $\zeta(t)$  has the behaviour of the above Pólya urn with weight function  $g$ . Let  $A$  be the event  $\{\lim_{t \rightarrow \infty} \zeta_k(t)/t = 1/n\}$ . Since  $\zeta_k^{(N)}(t)/t \rightarrow 1/n$  a.s., we have

$$P(A) \geq P(A | B_N)P(B_N) = P(B_N).$$

On the other hand, Lemma 2 implies that  $P(B_N) \rightarrow 1$  as  $N \rightarrow \infty$ , which finishes the proof.

## 4.6 Proof of Theorem 6

*Proof of Part 1) of Theorem 6.* Throughout the proof, denote the center of the star graph by  $n+1$  and all other vertices  $1, 2, \dots, n$ . We skip the trivial case, where  $\alpha < 0$  and  $\beta \leq 0$ .

We will show that if

$$\alpha < 0 < \beta, \text{ and } \alpha + \beta\sqrt{n} < 0, \quad (37)$$

then the stationary distribution is well defined. Let  $A_Q$  be the matrix determined by equation (17) in the case of the star graph with  $n+1$  vertices. Denote by  $D_n(\mu)$  be the characteristic polynomial of matrix  $A_Q$ . A direct computation gives the following recursive equation

$$D_n(\mu) = (-\alpha - \mu)D_{n-1}(\mu) - \beta^2(-\alpha - \mu)^{n-1}, \quad n \geq 1,$$

which yields that

$$D_n(\mu) = (-1)^{n+1}(\mu + \alpha)^{n-1}(\mu + \alpha + \beta\sqrt{n})(\mu + \alpha - \beta\sqrt{n}).$$

Thus,  $-\alpha > 0$  is the matrix eigenvalue of multiplicity  $n-1$  and  $-\alpha \pm \beta\sqrt{n}$  are eigenvalues of multiplicity 1. The eigenvalue  $-\alpha - \beta\sqrt{n} > 0$  is the minimal one (since  $\beta > 0$ ), hence  $A_Q$  is positive definite provided conditions (37) are satisfied. Positive definiteness of  $A_Q$  implies that  $Z_{\alpha, \beta, \Lambda} < \infty$  (as in the proof of Part 1) of Theorem 1). Therefore, the stationary distribution is well defined and the CTMC  $\xi(t)$  is ergodic.

Let us show that if  $\alpha < 0 < \beta$  and  $\alpha + \beta\sqrt{n} > 0$  then  $Z_{\alpha, \beta, \Lambda} = \infty$  and the stationary distribution is not defined. Start with noticing that  $(1, \dots, 1, \sqrt{n}) \in \mathbb{Z}_+^{n+1}$  is the eigenvector corresponding to the eigenvalue  $(-\alpha - \beta\sqrt{n})$ . Therefore, if  $\alpha + \beta\sqrt{n} \geq 0$  then the function  $\exp(-Q(\xi)/2)$  is not summable along the direction of this eigenvalue and, hence, the CTMC  $\xi(t)$  is not ergodic. Indeed, in this case, since  $\alpha < 0$ ,

$$Z_{\alpha, \beta, \Lambda} = \sum_{\xi \in \mathbb{Z}_+^{n+1}} e^{-Q(\xi)/2 - \frac{\alpha}{2} \sum_{i=1}^{n+1} \xi_i} \geq \sum_{\xi \in \mathbb{Z}_+^{n+1} \cap G} e^{-Q(\xi)/2}$$

where  $G = \{\xi : \xi_i = [\beta\xi_{n+1}/|\alpha|], i = 1, 2, \dots, n\}$  and  $[x]$  denotes the closest integer to  $x \in \mathbb{R}$ , so that  $|x - [x]| \leq 1/2$ . Using the expression (8) for  $Q(\xi)$  and the fact that  $\beta > \sqrt{n}|\alpha|$  we have

$$\begin{aligned} Z_{\alpha, \beta, \Lambda} &\geq \sum_{\xi \in \mathbb{Z}_+^{n+1} \cap G} \exp\left(-\frac{n|\alpha|}{8} + \frac{n\beta^2 - \alpha^2}{2|\alpha|} \xi_{n+1}^2\right) \\ &= e^{-\frac{n|\alpha|}{8}} \sum_{k=0}^{\infty} \exp\left(\frac{n\beta^2 - \alpha^2}{2|\alpha|} k^2\right) = \infty. \end{aligned}$$

*Proof of Part 2) of Theorem 6.* Observe that

$$\begin{aligned} U(i, \zeta) &= -|\alpha|\zeta_i + \beta\zeta_{n+1}, \quad i = 1, 2, \dots, n; \\ U(n+1, \zeta) &= -|\alpha|\zeta_{n+1} + \beta \sum_{i=1}^n \zeta_i. \end{aligned}$$

An easy calculation gives the following identity

$$(n\beta + |\alpha|)U(n+1, \zeta) + (\beta + |\alpha|) \sum_{i=1}^n U(i, \zeta) = (n\beta^2 - \alpha^2) S(\zeta) \quad (38)$$

where  $S$  is defined by (9), valid in the case of any star graph. Thus, if  $\alpha + \sqrt{n}\beta = 0$ ,

$$\begin{aligned} &(n\beta + |\alpha|)U(n+1, \zeta) + (\beta + |\alpha|) \sum_{i=1}^n U(i, \zeta) \\ &= \beta(1 + \sqrt{n}) \left( \sqrt{n}U(n+1, \zeta) + \sum_{i=1}^n U(i, \zeta) \right) = 0 \end{aligned}$$

which is equivalent to

$$\sqrt{n}U(n+1, \zeta) + \sum_{i=1}^n U(i, \zeta) = 0. \quad (39)$$

Given  $\xi$  denote

$$m_n = m_n(\xi) = \max_{i=1, \dots, n} \xi_i,$$

and

$$\tau_N = \min\{t : \max(\xi_{n+1}(t), \lfloor \sqrt{n}m_n(\xi(t)) \rfloor) = N\}$$

where  $\lfloor a \rfloor \leq a$  denotes the integer part of  $a$ . It is obvious that the Markov chain is explosive if and only if

$$\mathbb{P} \left( \lim_{N \rightarrow \infty} \tau_N < \infty \right) > 0,$$

but this cannot happen. Indeed, if  $\xi_{n+1} \geq \lfloor \sqrt{n}m_n \rfloor$  then

$$U_{n+1} = \beta(-\sqrt{n}\xi_{n+1} + \xi_1 + \dots + \xi_n) \leq \sqrt{n}\beta(-\xi_{n+1} + \sqrt{n}m_n) \leq 0,$$

and, on the other hand, if  $\xi_{n+1} < \lfloor \sqrt{n}m_n \rfloor$  then

$$U_k = \beta(-\sqrt{n}m_n + \xi_{n+1}) = \beta [(-\sqrt{n}m_n + \lfloor \sqrt{n}m_n \rfloor) - (\lfloor \sqrt{n}m_n \rfloor - \xi_{n+1})] < -\beta$$

for all  $k$  such that  $\xi_k = m_n$ . Therefore the waiting time  $\tau_{N+1} - \tau_N$  is stochastically larger than a certain exponentially distributed random variable which parameters depend only on  $n$  and  $\beta$  and hence the limit  $\lim_{N \rightarrow \infty} \tau_N$  is infinite with probability 1.

Now, let us prove transience of DTMC  $\zeta(t)$ . Recall that  $v = (1, \dots, 1, \sqrt{n}) \in \mathbb{Z}_+^{n+1}$  is the eigenvector corresponding to the eigenvalue  $(-\alpha - \beta\sqrt{n})$ . Define a function  $f$  as the scalar

product (in  $\mathbb{R}^{n+1}$ ) of vectors  $\zeta$  and  $v$ , i.e.  $f(\zeta) = \zeta_1 + \dots + \zeta_n + \sqrt{n}\zeta_{n+1}$ . For simplicity, denote  $f_t = f(\zeta(t))$ . We will show that there exists  $\varepsilon > 0$  such that for all  $\zeta$

$$\mathbb{E}[f_{t+2} - f_t \mid \zeta(t) = \zeta] \geq \varepsilon. \quad (40)$$

Since the function  $f$  is non-negative and has uniformly bounded jumps (as  $|f_{t+1} - f_t| \leq \sqrt{n}$ ) transience of  $\zeta(t)$  will follow from Theorem 2.2.7 in [4] with  $k(\alpha) \equiv 2$ .

To establish (40), observe that for  $\varepsilon \in [0, 1)$

$$\begin{aligned} & \mathbb{E}[f_{t+1} - f_t \mid \zeta(t) = \zeta] - \varepsilon \\ &= \frac{\sum_{i=1}^n e^{U(i, \zeta)} + \sqrt{n}e^{U(n+1, \zeta)} - \sum_{i=1}^n 1_{\{\zeta_i > 0\}} - \sqrt{n}1_{\{\zeta_{n+1} > 0\}}}{\sum_{i=1}^{n+1} [e^{U(i, \zeta)} + 1_{\{\zeta_i > 0\}}]} - \varepsilon \\ &= \frac{H(\zeta, \varepsilon)}{\sum_{i=1}^{n+1} [e^{U(i, \zeta)} + 1_{\{\zeta_i > 0\}}]} \end{aligned} \quad (41)$$

where

$$\begin{aligned} H(\zeta, \varepsilon) &= (1 - \varepsilon) \sum_{i=1}^n e^{U(i, \zeta)} + (\sqrt{n} - \varepsilon) e^{U(n+1, \zeta)} \\ &\quad - (1 + \varepsilon) \sum_{i=1}^n 1_{\{\zeta_i > 0\}} - (\sqrt{n} + \varepsilon) 1_{\{\zeta_{n+1} > 0\}}. \end{aligned}$$

From (39) and the inequality between the arithmetical and geometric means we have

$$\sum_{i=1}^n e^{U(i, \zeta)} \geq n \left[ \prod_{i=1}^n e^{U(i, \zeta)} \right]^{1/n} = ne^{-\frac{U(n+1, \zeta)}{\sqrt{n}}}$$

hence

$$\begin{aligned} \frac{H(\zeta, \varepsilon)}{1 - \varepsilon} &> \sum_{i=1}^n e^{U(i, \zeta)} + \frac{(\sqrt{n} - \sqrt{n}\varepsilon)}{1 - \varepsilon} e^{U(n+1, \zeta)} - \frac{1 + \varepsilon}{1 - \varepsilon} \sum_{i=1}^n 1_{\{\zeta_i > 0\}} \\ &\quad - \frac{\sqrt{n} + \sqrt{n}\varepsilon}{1 + \varepsilon} 1_{\{\zeta_{n+1} > 0\}} \\ &= \sum_{i=1}^n e^{U(i, \zeta)} + \sqrt{n}e^{U(n+1, \zeta)} - \frac{1 + \varepsilon}{1 - \varepsilon} (n + \sqrt{n}) =: \varphi_\varepsilon(u) \end{aligned}$$

where

$$\varphi_\varepsilon(u) = ne^{-u/\sqrt{n}} + \sqrt{n}e^u - \frac{1 + \varepsilon}{1 - \varepsilon} (n + \sqrt{n})$$

and  $u = U(n+1, \zeta) \in \mathbb{R}$ .

One can easily check that  $\varphi'_\varepsilon(0) = 0$  and  $\varphi''_\varepsilon(u) = e^{-u/\sqrt{n}} + \sqrt{n}e^u > 0$  for all  $u$ , therefore  $\varphi_\varepsilon(\cdot)$  attains its unique minimum at  $u = 0$ . If we set  $\varepsilon = 0$  we also have  $\varphi_0(0) = 0$  hence  $\varphi_0(u) \geq 0$ ,  $u \in \mathbb{R}$  implying that when  $\varepsilon = 0$  the LHS of (41) is always non-negative and  $f_t$  is thus a submartingale.

To show that it actually increases on average by at least  $\varepsilon > 0$  in *two* steps, note that  $|U(n+1, \zeta(t+1)) - U(n+1, \zeta(t))| \geq \beta > 0$  since  $\zeta(t+1)$  differs from  $\zeta(t)$  in one of the coordinates, and  $|\alpha| > \beta$ . Therefore,

$$\min \{|U(n+1, \zeta(t))|, |U(n+1, \zeta(t+1))|\} \geq \frac{\beta}{2}.$$

Without loss of generality, assume that it is  $u = U(n+1, \zeta(t))$  which has the property  $|u| \geq \beta/2$ . To guarantee that the LHS (41) is non-negative for some small  $\varepsilon > 0$  we will establish that

$$\inf_{u: |u| \geq \beta/2} \varphi_\varepsilon(u) = \min\{\phi_\varepsilon(-\beta/2), \phi_\varepsilon(\beta/2)\} > 0 \quad (42)$$

where the equality follows from the fact that  $\varphi_\varepsilon(u)$  is increasing for  $u > 0$  and decreasing for  $u < 0$ . However, since  $\varphi_0(\pm\beta/2)$  is strictly positive, as we established before, and  $\varphi_\varepsilon(u)$  is continuous in  $\varepsilon$ , by choosing  $\varepsilon > 0$  sufficiently small we can ensure (42) and hence (40) and transience.

*Proof of Part 3) of Theorem 6.* Recall from (38) that

$$(n\beta + |\alpha|)U(n+1, \zeta) + (\beta + |\alpha|) \sum_{i=1}^n U(i, \zeta) = (n\beta^2 - \alpha^2) S(\zeta),$$

where now  $n\beta^2 - \alpha^2 > 0$ , due to our assumption  $\beta > |\alpha|/\sqrt{n}$ . Hence, using the elementary fact that if  $a_1 + \dots + a_{n+1} = x$  then  $\max_i a_i \geq x/(n+1)$  we get that

$$\max_{i=1, \dots, n+1} U(i, \zeta(k)) \geq CS(\zeta(k))$$

and  $C > 0$  is some constant depending on  $n, \alpha$  and  $\beta$ .

At the same time, whenever any of the component of  $\zeta$  increases,  $S(\zeta(k))$  also increases by 1. For a positive integer  $y$  define  $\tau_y = \min\{t : S(\zeta(t)) \geq y\}$ . For each  $y \in \{1, 2, \dots\}$  the set of configurations of  $\zeta$  where  $S(\zeta) < y$  is finite, so with probability one at some point of time  $k$  the system will reach the state where  $S(\zeta(k)) \geq y$ , consequently  $\tau_y < \infty$  a.s. for all  $y$ . Hence we can define the events  $A_y =$  “there exists  $t \geq \tau_y$  such that some component decreases at time  $t$ ”. Then one can easily obtain the following bound

$$\mathbb{P}(A_y) \leq 1 - \prod_{k=y}^{\infty} \left(1 - \frac{n}{e^{\max_i U(i, \zeta(k), i)}}\right) \leq 1 - \prod_{k=y}^{\infty} \left(1 - \frac{n}{e^{Ck}}\right) \sim \frac{n}{1 - e^{-C}} \cdot e^{-Cy}$$

for large enough  $y$ . Since  $\sum_y e^{-Cy} < \infty$  by Borel-Cantelli lemma there will be a.s. a time  $y'$  for which no  $A_y$  ( $y \geq \tau_{y'}$ ) occurs and thus the only changes in the system are increases of the components; this also implies that for any integer  $k > \tau_{y'}$  we have  $\max_i U(i, \zeta(k)) \geq C(k - k')$ , thus ensuring that the CTMC  $\xi(t)$  explodes a.s., since the rates of jumps are bounded below by  $e^{C(k-k')}$ , the inverses of which are again summable.

Let us now observe the DTMC after time  $k'$  thus assuming only increases of the components, i.e.  $S(\zeta(k+1)) - S(\zeta(k)) = 1$  for all  $k \geq k'$ . Denote

$$z(k) = \sum_{i=1}^n \zeta_i(k) = S(\zeta(k)) - \zeta_{n+1}(k).$$

Since the probability that only the component at  $n + 1$  increases after time  $k$  equals

$$\prod_{l=k}^{\infty} \frac{e^{U(n+1, \zeta(k)) - |\alpha|(l-k)}}{e^{U(n+1, \zeta(k)) - |\alpha|(l-k)} + \sum_{i=1}^n e^{U(i, \zeta(k))}} = 0$$

on one hand, and the probability that the component at  $n + 1$  never increases after time  $k$  is equal to

$$\begin{aligned} & \prod_{l=k}^{\infty} \left( 1 - \frac{e^{U(n+1, \zeta(k))}}{e^{U(n+1, \zeta(k))} + \sum_{i=1}^n e^{U(i, \zeta(l))}} \right) \\ & \leq \prod_{l=k}^{\infty} \left( 1 - \frac{e^{U(n+1, \zeta(k))}}{e^{U(n+1, \zeta(k))} + n e^{\max_{i=1, \dots, n} U(i, \zeta(l))}} \right) \\ & = \prod_{l=k}^{\infty} \left( 1 - \left[ 1 + n \exp \left\{ (|\alpha| + 2\beta) \zeta_{n+1}(k) - \beta S(\zeta(l)) - |\alpha| \min_{i=1, \dots, n} \zeta_i(l) \right\} \right]^{-1} \right) \\ & \leq \prod_{l=k}^{\infty} C \cdot e^{-\beta l} = 0 \end{aligned}$$

on the other hand, we conclude that both  $\zeta_{n+1}(k) \rightarrow \infty$  and  $z(k) \rightarrow \infty$ .

Now consider the process  $\zeta(k)$  at those times  $k_1 < k_2 < \dots$  when one of the components in  $\{1, 2, \dots, n\}$  increases. It is easy to see that  $z(k_{i+1}) - z(k_i) = 1$  for all  $i$  and that one can couple the process  $(\zeta_1(k_i), \zeta_2(k_i), \dots, \zeta_n(k_i))$ ,  $i = 1, 2, \dots$ , with the generalized Pólya urn with  $n$  types of balls and the weight function  $g(x) = e^{\alpha x}$ . Since  $\alpha < 0$ , from, for example, a trivial comparison with the Friedman urn, we conclude that all  $\zeta_j(k_i)$ ,  $j = 1, \dots, n$  grow at the same speed, resulting in  $\zeta_j(k)/z(k) \rightarrow 1/n$ . Therefore, for any  $\epsilon > 0$  there is a (random) time  $k_1 \geq k'$  such that

$$\frac{1 - \epsilon}{n} \leq \min_{j=1, \dots, n} \frac{\zeta_j(k)}{z(k)} \leq \max_{j=1, \dots, n} \frac{\zeta_j(k)}{z(k)} \leq \frac{1 + \epsilon}{n} \text{ for all } k \geq k_1.$$

Once this being the case, the odds that at time  $k$  the component at  $n + 1$  grows (as opposed to a component at  $i$ ,  $i \in \{1, \dots, n\}$ ) lies in the interval

$$\left[ \frac{e^{-|\alpha|\zeta_{n+1} + \beta z}}{n e^{-|\alpha|(1-\epsilon)\frac{z}{n} + \beta \zeta_{n+1}}}, \frac{e^{-|\alpha|\zeta_{n+1} + \beta z}}{n e^{-|\alpha|(1+\epsilon)\frac{z}{n} + \beta \zeta_{n+1}}} \right] = [e^{zR_{-\epsilon} - L\zeta_{n+1} - \log(n)}, e^{zR_{+\epsilon} - L\zeta_{n+1} - \log(n)}]$$

where

$$R_{\pm\epsilon} = \beta + \frac{|\alpha|(1 \pm \epsilon)}{n}, \quad L = |\alpha| + \beta.$$

Let  $X(k) = z(k)R_{-\epsilon} - \zeta_{n+1}(k)L$ ,  $k = k_1, k_1 + 1, \dots$ . Then  $X(k)$  can be coupled with random walk  $Y(k)$  on  $[\log(np/(1-p)), +\infty)$  with the transitional probabilities

$$Y(k+1) = \begin{cases} Y(k) + R_{-\epsilon}, & \text{with probability } 1 - p; \\ \max \left\{ Y(k) - L, \log \left( \frac{np}{1-p} \right) \right\}, & \text{with probability } p, \end{cases}$$



in such a way that  $X(k) \leq Y(k)$ . By choosing  $p \in (0, 1)$  such that  $\mathbf{E}(Y(k+1) - Y(k)) = R_{-\epsilon}(1-p)Lp < 0$  (provided  $Y(k) \geq L + \log(np/(1-p))$ ) we ensure that  $\lim_{k \rightarrow \infty} Y(k)/k = 0$ , implying in turn that

$$\limsup_{k \rightarrow \infty} \frac{X(k)}{k} = \limsup_{k \rightarrow \infty} \frac{z(k)R_{-\epsilon} - \zeta_{n+1}(k)L}{k} \leq 0.$$

By the completely symmetric argument we also obtain

$$\liminf_{k \rightarrow \infty} \frac{z(k)R_{+\epsilon} - \zeta_{n+1}(k)L}{k} \geq 0.$$

Now, using the fact that  $z(k) + \zeta_{n+1}(k) = k + \text{const}$  for large  $k$ ,

$$\frac{R_{-\epsilon}}{L + R_{-\epsilon}} \leq \liminf_{k \rightarrow \infty} \frac{\zeta_{n+1}(k)}{k} \leq \limsup_{k \rightarrow \infty} \frac{\zeta_{n+1}(k)}{k} \leq \frac{R_{+\epsilon}}{L + R_{+\epsilon}}$$

Since  $\epsilon > 0$  is arbitrary and  $R_{+\epsilon} - R_{-\epsilon} \rightarrow 0$  as  $\epsilon \rightarrow 0$ , we get

$$\lim_{k \rightarrow \infty} \frac{\zeta_{n+1}(k)}{k} = \frac{\beta + |\alpha|/n}{\beta + |\alpha|/n + \beta + |\alpha|} = \frac{n\beta + |\alpha|}{2n\beta + (n+1)|\alpha|}$$

and, as a consequence,

$$\lim_{k \rightarrow \infty} \frac{\zeta_i(k)}{k} = \frac{\beta + |\alpha|}{2n\beta + (n+1)|\alpha|} \text{ for } i = 1, 2, \dots, n.$$

Finally, we also conclude that all the components of the CTMC  $\xi$  actually explode simultaneously.

*Proof of Part 4) of Theorem 6.* The case *i)* of the theorem is covered by Theorem 2, and the case *ii)* is covered by Theorem 3, since a star graph does not have triangles.

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